Nonlinear Systems and Control

Lecture 9

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Master Course in Electronic and Communication Engineering
Credits (3/0/3)

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Autonomous Systems with Inputs and Outputs: Definition

State Equations

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

- State $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$
- f is locally Lipschitz, h continuous
- $f(0,0) = 0$ and $h(0,0) = 0$

Transfer Block

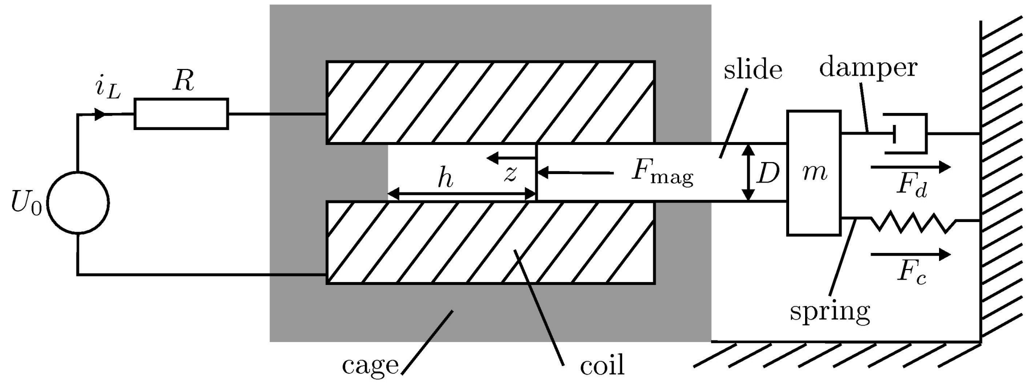
Gap 1

Goal

- Study Lyapunov stability and input/output stability of such systems
- Introduce concept of passivity to describe "energy storage" of such systems
- Develop feedback control methods to achieve stability

Application of Lyapunov's Stability Theory: Magnetic Valve

Simple Electromagnetic Valve



- Slide (mass m) moves according to magnetic force F_{mag}
- Spring force $F_c = c \cdot z$
- Damper force $F_d = d \cdot \dot{z}$
- Inductance $L(z) = \dfrac{\mu_0 N^2 D^2 \pi}{4(h-z)}$ (coil: N windings, $\mu_0 = 4\pi \cdot 10^{-7} \dfrac{Vs}{Am}$)

Application of Lyapunov's Stability Theory: Magnetic Valve

Force balance

Gap 2

Application of Lyapunov's Stability Theory: Magnetic Valve

State Equations

- State Variables: $x_1 = z$, $x_2 = \dot{z}$, $x_3 = i_{\mathrm{L}}$, input: $u = U_0$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}\left(\frac{1}{2}\frac{\partial L(x_1)}{\partial x_1}x_3^2 - cx_1 - dx_2\right) \\
\dot{x}_3 &= \frac{1}{L(z)}\left(u - Rx_3 - \frac{\partial L(x_1)}{\partial x_1}x_3x_2\right) \\
y &= x_3
\end{aligned}$$

Energy Function

$$V(x) = \frac{1}{2}(cx_1^2 + mx_2^2 + L(x_1)x_3^2)$$

Autonomous Systems with Inputs and Outputs Application of the Lyapunov Stability Theory Input/Output Stability Passivity

Application of Lyapunov's Stability Theory: Magnetic Valve

Stability Analysis

$$\dot{V}(x) = \begin{bmatrix} cx_1 + \frac{1}{2}\frac{\partial L(x_1)}{\partial x_1}x_3^2 & mx_2 & L(x_1)x_3 \end{bmatrix} f(x, u)$$

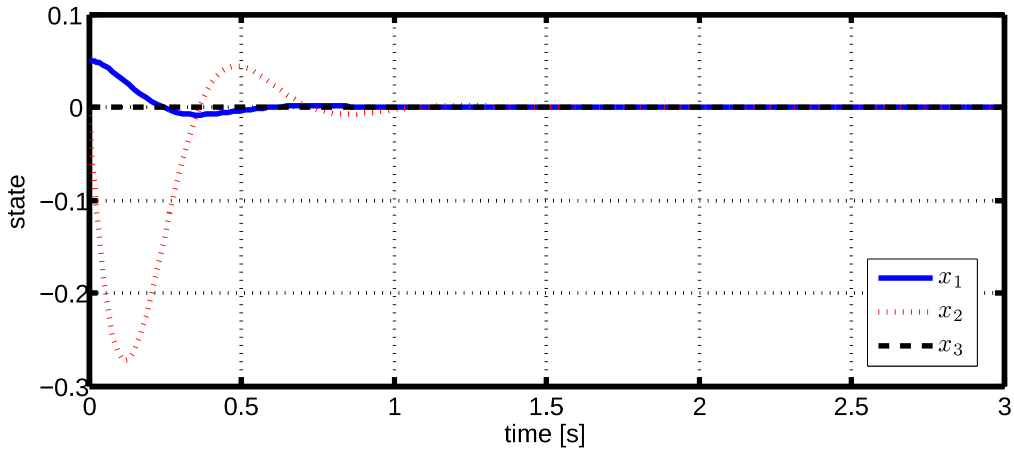
Gap 3

Equilibrium Point for $u = 0$

$$x_1 = x_2 = x_3 = 0$$

$\Rightarrow$ Asymptotic stability according to Corollary to La Salle's Theorem

Application of Lyapunov's Stability Theory: Magnetic Valve



Parameters

- $m = 0.5\,\text{kg},\ R = 10\,\Omega,\ d = 10\,\text{N·s/m},\ c = 100\,\text{N/m}$
- $h = 0.1\,\text{m},\ D = 0.05\,\text{m},\ N = 20$
- Initial condition: $x_1 = 0.05\,\text{m},\ x_2 = x_3 = 0$

$\Rightarrow$ What to do if input $u \neq 0$ or input u is not constant?

Input/Output Stability: Notation

Linear Signal Spaces

- $\mathcal{L}_2$-space: Set of all square integrable vector functions u

$$\int_0^\infty u^T(\tau)u(\tau)d\tau < \infty$$

- $\mathcal{L}_2$-norm

$$||u||_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(\tau)u(\tau)d\tau}$$

- Input/Output Mapping

$$H : u \rightarrow y : \mathcal{L}_2^m \rightarrow \mathcal{L}_2^p$$

Illustration

Gap 4

- u: input, y: output

Input/Ouput Stability: Definition

Definition (Finite-Gain Input/Output Stability)

An input/output mapping $H : \mathcal{L}_2^m \to \mathcal{L}_2^p$ is finite-gain $\mathcal{L}_2$-stable if there exists a $\gamma > 0$ and $\beta \geq 0$ such that

$$||y_t||_{\mathcal{L}_2} = ||Hu_t||_{\mathcal{L}_2} \leq \gamma ||u_t||_{\mathcal{L}_2} + \beta$$

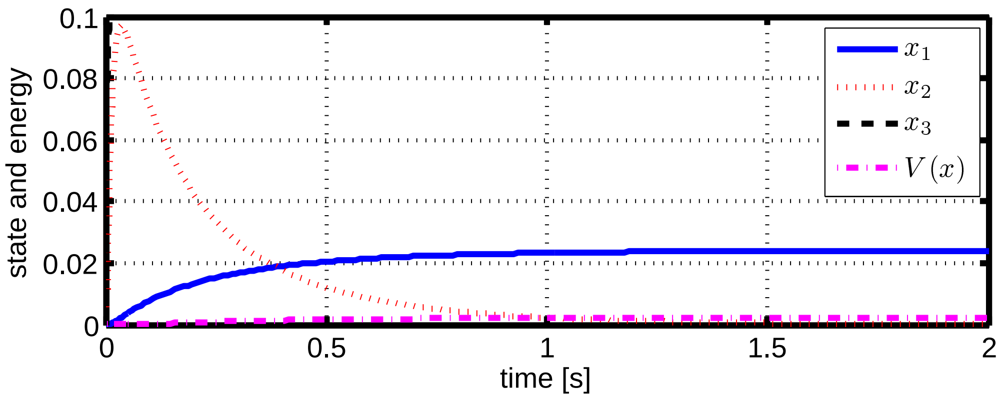
for all $u \in \mathcal{L}_2^m$ and $t \in [0, \infty)$. Here, $||y_t||_{\mathcal{L}_2} = \sqrt{\int_0^t y^T(\tau) y(\tau) d\tau}$.

Gap 5

Input/Output Stability: Magnetic Valve

- Consider input $u = U_0 = 10 \Rightarrow ||u_t||^2_{\mathcal{L}_2} = \int_0^t (10)^2 d\tau = 100t$
- Observe output $y = x_3 = 0 \to ||y_t||^2_{\mathcal{L}_2} = 0$

  $\Rightarrow$ This input/output pair fulfills input/output stability condition



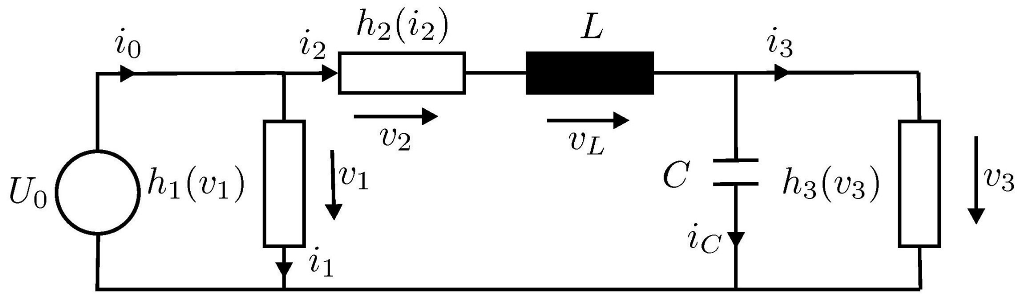
Question

$\Rightarrow$ How to verify input/output stability for all input/output pairs?

$\Rightarrow$ Study "Passivity" of autonomous systems with inputs and outputs

Passivity: Electric Circuit

RLC-Circuit



- Input voltage U_0
- Current i_0
- Inductance L
- Capacitance C
- Resistor characteristics: $i_1 = h_1(v_1)$, $v_2 = h_2(i_2)$, $i_3 = h_3(i_3)$

Passivity: Electric Circuit

State Equations

- State variables: $x_1 = i_2$, $x_2 = u_3$, input: $u = U_0$, output $y = i_0$

$$\dot{x}_1 = \frac{1}{L}(u - h_2(x_1) - x_2)$$
$$\dot{x}_2 = \frac{1}{C}(x_1 - h_3(x_2))$$
$$y = x_1 + h_1(u)$$

System Energy

$$V(x) = \frac{1}{2}(Lx_1^2 + Cx_2^2)$$

Passivity: Electric Circuit

Stability Analysis

$$\dot{V}(x) = \begin{bmatrix} Lx_1 & Cx_2 \end{bmatrix} f(x,u) =$$

Gap 6

$$\Rightarrow yu = \dot{V}(x) + uh_1(u) + x_1 h_2(x_1) + x_2 h_3(x_2)$$

Interpretation

- yu is the absorbed power of the network
- $\dot{V}(x)$ is the stored power in the network

$\Rightarrow$ System is called "passive" if $yu \geq \dot{V}(x)$

Passivity: Different Cases

Case 1: $h_1 = h_2 = h_3 = 0$
$\Rightarrow$ **Circuit is lossless**

Case 2: $x_1 h_2(x_1) + x_2 h_3(x_2)$ is positive semi-definite and $uh_1(u)$ is positive definite
$\Rightarrow$ **Circuit is input strictly passive**

Case 3: $h_1 = 0$, h_3 is positive semi-definite and $yh_2(y)$ is positive definite
$\Rightarrow$ **Circuit is output strictly passive**

Case 4: $uh_1(u)$ is positive definite and $x_1 h_2(x_1) + x_2 h_3(x_2)$ is positive definite
$\Rightarrow$ **Circuit it state strictly passive**

Gap 7

Passivity: Definition

Definition (Passivity)

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = h(x,u) \qquad y \in \mathbb{R}^m$$

is said to be passive if there exists a continuously differentiable positive semi-definite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x,u), \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n$$

The system is called

- lossless: $u^T y = \dot{V}(x)$
- input strictly passive: $u^T y \geq \dot{V}(x) + u^T \varphi(u)$ and $u^T \varphi(u)$ pos. def.
- output strictly passive: $u^T y \geq \dot{V}(x) + y^T \varphi(y)$ and $y^T \varphi(y)$ pos. def.
- strictly passive: $u^T y \geq \dot{V}(x) + \varphi(x)$ and $\varphi(x)$ pos. def.

Passivity: Input/Output Stability

Lemma (Input/Output Stability)

If the system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = h(x,u) \qquad y \in \mathbb{R}^m$$

is output strictly passive with $u^T y \geq \dot{V}(x) + \delta y^T y$ for some $\delta > 0$, then it is finite-gain $\mathcal{L}_2$-stable and its $\mathcal{L}_2$-gain is less than or equal to $1/\delta$.

Magnetic Valve Example

Gap 8

⇒ **System is input/output stable**

⇒ $\mathcal{L}_2$**-gain is less or equal to** $1/R$

Passivity: Lyapunov Stability

Lemma (Lyapunov Stability)

Consider the system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = h(x,u) \qquad y \in \mathbb{R}^m$$

with a positive definite storage function $V(x)$. If the system is passive, then the origin of $\dot{x} = f(x,0)$ is stable.

Magnetic Valve Example

Gap 9

⇒ **Origin of $\dot{x} = f(x,0)$ is stable according to Lyapunov**

Passivity: Lyapunov Asymptotic Stability

Definition (Zero-state Detectability)

The system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = h(x,u) \qquad y \in \mathbb{R}^m$$

is locally (globally) zero-state detectable if there exists a neighborhood $\mathcal{D}$ ($\mathcal{D} = \mathbb{R}^n$) of $x = 0$ such that for any trajectory starting in $\mathcal{D}$

$$u \equiv 0 \text{ and } y \equiv 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$$

Lemma (Lyapunov Asymptotic Stability)

Consider the system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = h(x,u) \qquad y \in \mathbb{R}^m$$

with a positive definite storage function $V(x)$. If the system is strictly passive or output strictly passive and zero-state detectable, then the origin of $\dot{x} = f(x,0)$ is asymptotically stable.

Passivity: Asymptotic Stability of Magnetic Valve

Corollary to La Salle's Theorem

Gap 10

Passivity: Asymptotic Stability of Magnetic Valve

Corollary to La Salle's Theorem

Gap 11

$\Rightarrow$ **Origin of magnetic valve is globally asymptotically stable**