Nonlinear Systems and Control

Lecture 8

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Credits (3/0/3)

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Invariance Principle: Motivation

Asymptotic Stability Theorem: Shortcomings

- Requires that $\dot{V}(x)$ is negative definite
- Analysis assumes that $x_1, x_2, \ldots, x_n$ are independent variables
- In reality, $x_1, x_2, \ldots, x_n$ are related by the nonlinear system equations $\dot{x} = f(x)$

$\Rightarrow$ **Analysis can be overly conservative**

Illustration: Pendulum with Friction

Gap 1

Invariance Principle: Motivation

Illustration: Pendulum with Friction

Gap 2

Invariance Principle: Set Definitions

Definition (Invariant Set)

A set M is said to be an *invariant set* with respect to the autonomous system $\dot{x} = f(x)$ if

$$x(t_0) \in M \Rightarrow x(t) \in M \text{ for all } t \in \mathbb{R}$$

Remark

- Trajectories that start in an invariant set M stay in M for all times

Example

- Each equilibrium point is an invariant set

Illustration

Gap 3

Invariance Principle: Set Definitions

Definition (Limit Set)

Let $x(t)$ be a trajectory of an autonomous system $\dot{x} = f(x)$. The set N is called the *limit set* of $x(t)$ if for any point $p \in N$, there exists a sequence $\{t_n\}$ of times with $t_n \in [0, \infty)$ and such that

$$\lim_{n \to \infty} ||x(t_n) - p|| = 0$$

Illustration

Gap 4

Invariance Principle: Relevant Properties

Lemma (Boundedness)

If the solution $x(t)$ of the autonomous system $\dot{x} = f(x)$ for some initial condition $x(t_0)$ is bounded for all $t > t_0$, then its limit set N is (i) bounded, (ii) closed and (iii) non-empty. Moreover, $\lim_{t \to \infty} x(t) \in N$.

Lemma (Invariance)

The limit set N of a solution $x(t)$ of the autonomous system $\dot{x} = f(x)$ for some initial condition $x(t_0)$ is invariant with respect to $\dot{x} = f(x)$.

Illustration

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Invariance Principle: La Salle's Theorem

Theorem (La Salle)

Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function and assume that

(i) $M \subset \mathcal{D}$ *is a compact set invariant with respect to* $\dot{x} = f(x)$

(ii) $\dot{V}(x) \leq 0$ *for all* $x \in M$

(iii) $E : \{x \in M | \dot{V}(x) = 0\}$ *contains all points in* M *where* $\dot{V}(x) = 0$

(iv) N *is the largest invariant set in* E

Then, any solution starting in M *approaches* N *as* $t \to \infty$

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Invariance Principle: La Salle's Theorem

Proof

Gap 7

Invariance Principle: La Salle's Theorem

Proof

Gap 8

Invariance Principle: Discussion

Extensions of Lyapunov Stability Theory

- $V(x)$ only needs to be continuously differentiable but not positive definite
- La Salle's theorem also applies to limit sets and not only to equilibrium points

  $\Rightarrow$ **La Salle for example also captures limit cycles!**

Verification of the Conditions

(i) M exists for example if $V(x)$ is positive definite and $\dot{V}(x) \leq 0$
(ii) Straightforward computation of $\dot{V}(x)$
(iii) Straightforward computation by inspection of $\dot{V}(x)$
(iv) Determine all points in E that fulfill $\dot{x} = f(x)$

Invariance Principle: Example

Example for La Salle's Theorem

$$\dot{x}_1 = x_2 + x_1(\beta^2 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(\beta^2 - x_1^2 - x_2^2), \quad \beta \in \mathbb{R}$$

Equilibrium Point

$$x_1 = x_2 = 0$$

Limit Cycle

$$x_1^2 + x_2^2 = \beta^2$$

System Equations on Limit Cycle

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

Illustration

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Invariance Principle: Example

Example for La Salle's Theorem

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Invariance Principle: Corollary

Corollary (Equilibrium Point)

The equilibrium point $x = 0$ of the autonomous system $\dot{x} = f(x)$ is asymptotically stable if there exists a function $V : \mathcal{D} \to \mathbb{R}$ such that

(i) *$V(x)$ is positive definite for all $x \in \mathcal{D}$ and $0 \in \mathcal{D}$*

(ii) *$\dot{V}(x)$ is negative semi-definite in a bounded region $M \subseteq \mathcal{D}$*

(iii) *$\dot{V}(x)$ does not vanish identically along any trajectory in M other than the null solution $x = 0$*

Relation to La Salle's Theorem

- (i) and (ii) in the corollary imply (i) and (ii) in La Salle's theorem
- (iii) in the corollary implies that N only contains $x = 0$ in La Salle's theorem

  $\Rightarrow$ **Convergence to the equilibrium point**

Invariance Principle: Example

Pendulum with Friction

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Topics Covered up to Now

Overview

- Existence of solutions to nonlinear ordinary differential equations
- Equilibrium points
- Analysis of nonlinear systems
  - Stability analysis by linearization
  - Phase plane analysis
  - Limit cycles
- Describing functions and harmonic balance
- Lyapunov stability theorems
- Invariance principle and La Salle's theorem
- Extensions to stability analysis
  - Global asymptotic stability
  - Region of Attraction
  - Instability