Nonlinear Systems and Control

Lecture 7

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Master Course in Electronic and Communication Engineering
Credits (3/0/3)

Webpage: http://ECE564.cankaya.edu.tr

Variable Gradient Method: Procedure

Determine Gradient of $V(x)$

$$\nabla V(x) = \frac{\partial V(x)}{\partial x} = g(x) = \begin{bmatrix} g_1(x) & \cdots & g_n(x) \end{bmatrix}$$

- Guess structure of $g(x)$ with free parameters
- Evaluate $\dot{V}(x) = \nabla V(x) f(x) = g(x) f(x)$ and choose free parameters such that $\dot{V}(x)$ is negative (semi-) definite

Compute $V(x)$

- Solve partial differential equation $\frac{\partial V(x)}{\partial x_1} = g_1(x) \quad \cdots \quad \frac{\partial V(x)}{\partial x_n} = g_n(x)$

Condition for $g(x)$

$$\begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n(x)}{\partial x_1} & \cdots & \frac{\partial g_n(x)}{\partial x_n} \end{bmatrix} \text{ must be symmetric}$$

Variable Gradient Method: Example

Example

Gap 1

Variable Gradient Method: Example

Example

Gap 2

Region of Attraction: Motivation

Lyapunov Asymptotic Stability

- Domain $\mathcal{D} \subset \mathbb{R}^n$
- $V(x)$ is positive definite in $\mathcal{D}$
- $\dot{V}(x)$ is negative definite in $\mathcal{D}$
  $\Rightarrow$ No global asymptotic stability
  $\Rightarrow$ Existence of subset of $\mathcal{D}$ where trajectories converge

Illustration

Gap 3

Definition (Region of Attraction)

Consider an autonomous nonlinear system $\dot{x} = f(x)$ with an asymptotically stable equilibrium point x_{e} and denote the trajectory with initial condition x_0 as $x(t, x_0)$. The region of attraction of x_{e} is

$$\mathcal{R}_{\mathrm{A}} = \{x_0 \in \mathcal{D} | x(t, x_0) \to x_{\mathrm{e}} \text{ as } t \to \infty\}$$

Region of Attraction: Theorem

Theorem (Region of Attraction)

Consider an autonomous nonlinear system $\dot{x} = f(x)$ with an asymptotically stable equilibrium point x_{e}. Let $V : \mathcal{D} \to \mathbb{R}$ be continuously differentiable. If

(i) *$M \subset \mathcal{D}$ is a compact invariant set with $x_{\mathrm{e}} \in M$*

(ii) *$\dot{V}(x) < 0$ for $x \in M - \{x_{\mathrm{e}}\}$ and $\dot{V}(x_{\mathrm{e}}) = 0$*

then, $M \subseteq \mathcal{R}_{\mathrm{A}}$.

Estimation of Region of Attraction

- Find region $\mathcal{D}$ where $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite
- Determine $c = \min_{x \in \delta\mathcal{D}} V(x)$ on border $\delta\mathcal{D}$ of region $\mathcal{D}$
  $\Rightarrow$ Choose $M = \{x \in \mathcal{D} | V(x) \leq c\}$

Illustration

Gap 4

Region of Attraction: Example

Example

Gap 5

Variable Gradient Method Region of Attraction Instability

Region of Attraction: Example

Example

Gap 6

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Instability: Theorem

Theorem (Chetaev)

Consider an autonomous nonlinear system $\dot{x} = f(x)$ with the equilibrium point $x_{\mathrm{e}} = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be such that

(i) $V(0) = 0$ *and there is* $x_0 \in \mathbb{R}^n$, *arbitrarily close to* x_{e}, *and* $V(x_0) > 0$

(ii) $\dot{V}(x) > 0$ *for all* $x \in U = \{x \in \mathcal{D}| \;\; ||x|| \leq \epsilon \; (\epsilon > 0)$ *and* $V(x) > 0\}$

Then, x_{e} *is instable.*

Explanation

- Trajectories that start arbitrarily close to $x_{\mathrm{e}} = 0$ diverge from the origin
  $\Rightarrow$ Instability of x_{e}

Illustration

Gap 7

Instability: Example

Example

Gap 8