Nonlinear Systems and Control

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Lyapunov Stability Theory for Autonomous Systems

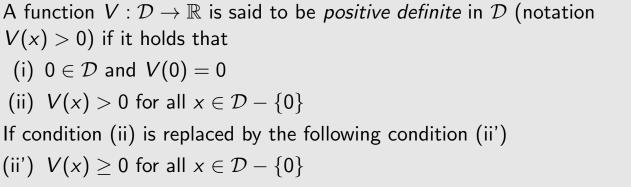
Lyapunov Stability Theory: Preliminaries

Goal:

Analyze stability properties of equilibrium points of autonomous systems

 \Rightarrow Lyapunov stability theory

Properties of Functions Definition (Positive Definite Functions)



then V is called *positive semi-definite* (notation $V(x) \ge 0$).

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Global Asymptotic Stability

Gap 1

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Lyapunov Stability Theory: Preliminaries

Definition (Negative Definite Functions)

A function $V : \mathcal{D} \to \mathbb{R}$ is called *negative definite (semi-definite)* (notation V(x) < 0 ($V(x) \le 0$)) if -V is positive definite (semi-definite).

Examples

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Lyapunov Stability Theory for Autonomous Systems

Global Asymptotic Stability

Lyapunov Stability Theory: Stability Theorem

Theorem (Lyapunov Stability Theorem)

Let x = 0 be an equilibrium point of $\dot{x} = f(x)$ with $f : \mathcal{D} \to \mathbb{R}^n$ and let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that V(x) is positive definite and $\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x)$ is negative semi-definite. Then x = 0 is stable.

Remarks

- Sufficient condition for stability of the equilibrium point $x_{\rm e}=0$
- V is called a Lyapunov candidate
- V is called a Lyapunov functions if it fulfills the stability theorem
- Instability of x_e does not follow if a Lyapunov candidate does not fulfill the stability theorem

Lyapunov Stability Theory: Stability Theorem

Proof

Gap 2

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Gap 3

Lyapunov Stability Theory: Stability Theorem

Proof

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Lyapunov Stability Theory: Example

Pendulum without Friction

• Energy function is good Lyapunov candidate

$$E = \frac{1}{2}m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l(1 - \cos \theta)$$

$$\Rightarrow V(x) = \frac{1}{2}m \cdot l^2 \cdot x_2^2 + m \cdot g \cdot l(1 - \cos x_1)$$

Stability

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Gap 4

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Lyapunov Stability Theory for Autonomous Systems

Lyapunov Stability Theory: Example

Pendulum with Friction

Gap 5

Lyapunov Stability Theory: Asymptotic Stability

Theorem

Let x = 0 be an equilibrium point of $\dot{x} = f(x)$ with $f : \mathcal{D} \to \mathbb{R}^n$ and let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that V(x) is positive definite and $\dot{V}(x)$ is negative definite. Then x = 0 is asymptotically stable.

Proof	
	Gap 6
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Lyapunov Stability Theory for Autonomous Systems

Lyapunov Stability Theory: Verification

Proof (Ctnd)

Gap 7

Global Asymptotic Stability

Verification of Lyapunov Stability

- Find V(x) > 0
 - \Rightarrow Easy task since independent of f
- Fulfill $\dot{V}(x) \le 0$ (or $\dot{V}(x) < 0$) \Rightarrow Difficult task since dependent on f
- We will study methods to find Lyapunov functions

Lyapunov Stability Theory: Example

Pendulum with Friction

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Lyapunov Stability Theory for Autonomous Systems

Global Asymptotic Stability

Global Asymptotic Stability: Definition

Motivation

- Consider an asymptotically stable equilibrium point
- Question: From which initial states do trajectories converge to equilibrium point?

<u>Illustration</u>

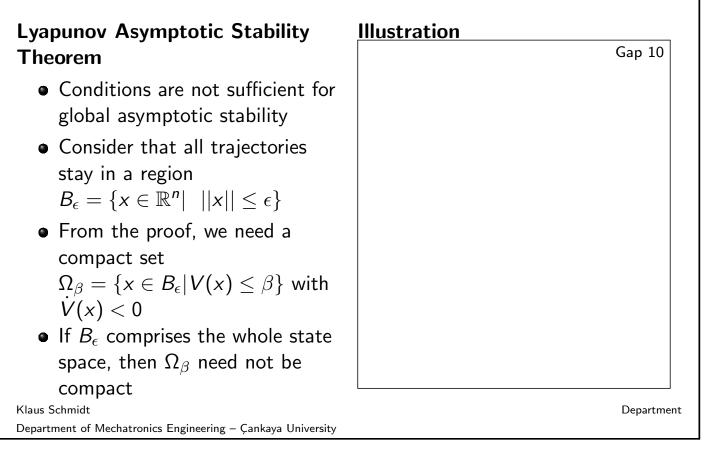
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Definition (Global Asymptotic Stability)

Consider an autonomous nonlinear system $\dot{x} = f(x)$ with an equilibrium point $x_e = 0$. The equilibrium point is said to be globally asymptotically stable (asymptotically stable in the large) if it is asymptotically stable and every trajectory starting in \mathbb{R}^n converges to x_e for $t \to \infty$.

Global Asymptotic Stability: Explanation



Lyapunov Stability Theory for Autonomous Systems

Global Asymptotic Stability

Global Asymptotic Stability: Counterexample

Example

Gap 11 **Problem** $\Rightarrow V(x)$ decreases although ||x|| becomes unbounded

Global Asymptotic Stability: Theorem

Definition

Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function. Then, V(x) is said to be *radially unbounded* if $V(x) \to \infty$ as $||x|| \to \infty$

Theorem (Global Asymptotic Stability)

Consider the autonomous nonlinear system $\dot{x} = f(x)$ with the equilibrium point $x_e = 0$. Let $\mathcal{D} = \mathbb{R}^n$ and let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

- (i) V(x) is positive definite
- (ii) $\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x)$ is negative definite
- (iii) V(x) is radially unbounded
- Then, $x_e = 0$ is globally asymptotically stable.

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Global Asymptotic Stability

Global Asymptotic Stability: Example

Example

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