Nonlinear Systems and Control

Lecture 6

Associate Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineering – Çankaya University

Master Course in Electronic and Communication Engineering
Credits (3/0/3)

Webpage: http://ECE564.cankaya.edu.tr

Lyapunov Stability Theory: Preliminaries

Goal:

- Analyze stability properties of equilibrium points of autonomous systems
  $\Rightarrow$ Lyapunov stability theory

Properties of Functions

Definition (Positive Definite Functions)

A function $V : \mathcal{D} \to \mathbb{R}$ is said to be *positive definite* in $\mathcal{D}$ (notation $V(x) > 0$) if it holds that

(i) $0 \in \mathcal{D}$ and $V(0) = 0$

(ii) $V(x) > 0$ for all $x \in \mathcal{D} - \{0\}$

If condition (ii) is replaced by the following condition (ii')

(ii') $V(x) \geq 0$ for all $x \in \mathcal{D} - \{0\}$

then V is called *positive semi-definite* (notation $V(x) \geq 0$).

Lyapunov Stability Theory: Preliminaries

Definition (Negative Definite Functions)

A function $V : \mathcal{D} \to \mathbb{R}$ is called *negative definite (semi-definite)* (notation $V(x) < 0$ ($V(x) \leq 0$)) if $-V$ is positive definite (semi-definite).

Examples

Gap 1

Lyapunov Stability Theory: Stability Theorem

Theorem (Lyapunov Stability Theorem)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$ with $f : \mathcal{D} \to \mathbb{R}^n$ and let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that $V(x)$ is positive definite and $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x)$ is negative semi-definite. Then $x = 0$ is stable.

Remarks

- Sufficient condition for stability of the equilibrium point $x_{\mathrm{e}} = 0$
- V is called a *Lyapunov candidate*
- V is called a *Lyapunov functions* if it fulfills the stability theorem
- Instability of x_{e} does not follow if a Lyapunov candidate does not fulfill the stability theorem

Lyapunov Stability Theory: Stability Theorem

Proof

Gap 2

Lyapunov Stability Theory: Stability Theorem

Proof

Gap 3

Lyapunov Stability Theory: Example

Pendulum without Friction

- Energy function is good Lyapunov candidate

$$E = \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l(1 - \cos\theta)$$

$$\Rightarrow V(x) = \frac{1}{2} m \cdot l^2 \cdot x_2^2 + m \cdot g \cdot l(1 - \cos x_1)$$

Stability

Gap 4

Lyapunov Stability Theory: Example

Pendulum with Friction

Gap 5

Lyapunov Stability Theory: Asymptotic Stability

Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$ with $f : \mathcal{D} \rightarrow \mathbb{R}^n$ and let $V : \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function such that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. Then $x = 0$ is asymptotically stable.

Proof

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Klaus Schmidt
Department of Mechatronics Engineering – Çankaya University

Lyapunov Stability Theory: Verification

Proof (Ctnd)

Gap 7

Verification of Lyapunov Stability

- Find $V(x) > 0$
  $\Rightarrow$ Easy task since independent of f
- Fulfill $\dot{V}(x) \leq 0$ (or $\dot{V}(x) < 0$)
  $\Rightarrow$ Difficult task since dependent on f
- We will study methods to find Lyapunov functions

Klaus Schmidt
Department of Mechatronics Engineering – Çankaya University

Lyapunov Stability Theory: Example

Pendulum with Friction

Gap 8

Global Asymptotic Stability: Definition

Motivation

- Consider an asymptotically stable equilibrium point
- Question: From which initial states do trajectories converge to equilibrium point?

Illustration

Gap 9

Definition (Global Asymptotic Stability)

Consider an autonomous nonlinear system $\dot{x} = f(x)$ with an equilibrium point $x_e = 0$. The equilibrium point is said to be globally asymptotically stable (asymptotically stable in the large) if it is asymptotically stable and every trajectory starting in $\mathbb{R}^n$ converges to x_e for $t \to \infty$.

Global Asymptotic Stability: Explanation

Lyapunov Asymptotic Stability Theorem

- Conditions are not sufficient for global asymptotic stability
- Consider that all trajectories stay in a region $B_\epsilon = \{x \in \mathbb{R}^n| \ \ ||x|| \leq \epsilon\}$
- From the proof, we need a compact set $\Omega_\beta = \{x \in B_\epsilon | V(x) \leq \beta\}$ with $\dot{V}(x) < 0$
- If B_ϵ comprises the whole state space, then Ω_β need not be compact

Illustration

Gap 10

Global Asymptotic Stability: Counterexample

Example

Gap 11

Problem

$\Rightarrow$ $V(x)$ decreases although $||x||$ becomes unbounded

Global Asymptotic Stability: Theorem

Definition

Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function. Then, $V(x)$ is said to be *radially unbounded* if $V(x) \to \infty$ as $||x|| \to \infty$

Theorem (Global Asymptotic Stability)

Consider the autonomous nonlinear system $\dot{x} = f(x)$ with the equilibrium point $x_{\mathrm{e}} = 0$. Let $\mathcal{D} = \mathbb{R}^n$ and let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

(i) $V(x)$ *is positive definite*

(ii) $\dot{V}(x) = \dfrac{\partial V(x)}{\partial x} f(x)$ *is negative definite*

(iii) $V(x)$ *is radially unbounded*

Then, $x_{\mathrm{e}} = 0$ is globally asymptotically stable.

Global Asymptotic Stability: Example

Example

Gap 12