Nonlinear Systems and Control

Lecture 5

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Master Course in Electronic and Communication Engineering
Credits (3/0/3)

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Stability: Basic Definition

System under Consideration

- Autonomous system: $\dot{x} = f(x)$, $f : \mathcal{D} \to \mathbb{R}^n$
- Domain $\mathcal{D} \subseteq \mathbb{R}^n$ is an open and connected set
- f is locally Lipschitz-continuous
- $x = x_\mathrm{e}$ is an equilibrium point: $f(x_\mathrm{e}) = 0$

Definition (Stability)

The equilibrium point $x = x_\mathrm{e}$ of the autonomous system $\dot{x} = f(x)$ is said to be stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$||x(t_0) - x_\mathrm{e}|| < \delta \Rightarrow \forall t \geq t_0 : ||x(t) - x_\mathrm{e}|| < \epsilon$$

$\Rightarrow$ If a trajectory starts close enough to the equilibrium point, then it will remain in a bounded neighborhood of the equioibrium point

Stability: Basic Definition

Phase Plane Illustration

Gap 1

Stability: Convergence

Definition (Convergence)

The equilibrium point $x = x_{\mathrm{e}}$ of the autonomous system $\dot{x} = f(x)$ is said to be *convergent* if there is a $\delta_1 > 0$ such that

$$||x(t_0) - x_{\mathrm{e}}|| < \delta_1 \Rightarrow \lim_{t\to\infty} x(t) = x_{\mathrm{e}}$$

Phase Plane Illustration

Gap 2

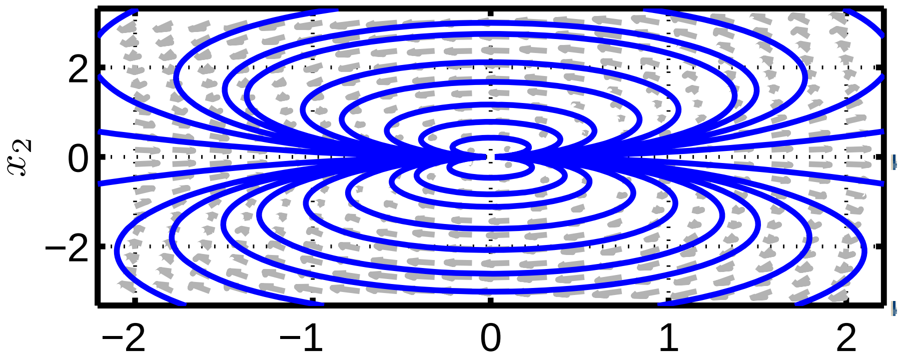
Stability: Convergence

Remark

- Convergence $\not\Rightarrow$ stability
- "Butterfly system"

$$\dot{x}_1 = x_1^2 - x_2^2$$
$$\dot{x}_2 = 2x_1x_2$$

$\Rightarrow$ Not all trajectories stay in a bounded neighborhood



Stability: Asymptotic Stability

Definition (Asymptotic Stability)

The equilibrium point $x = x_e$ of the autonomous system $\dot{x} = f(x)$ is said to be *asymptotically stable* if it is both stable and convergent.

Remark

- Asymptotic stability is desired in many applications
- Disadvantage: No information about rate of convergence
  $\Rightarrow$ Slow convergence is usually undesired

Time Evolution

Gap 3

Stability: Exponential Stability

Definition (Exponential Stability)

The equilibrium point $x = x_\mathrm{e}$ of the autonomous system $\dot{x} = f(x)$ is said to be locally *exponentially stable* if there exist $\alpha, \lambda > 0$ such that

$$\text{for all } t \geq 0 : ||x(t) - x_\mathrm{e}|| \leq \alpha ||x(t_0) - x_\mathrm{e}|| e^{-\lambda t}$$

whenever $||x(t_0) - x_\mathrm{e}|| < \delta$. It is said to be globally exponentially stable if the above condition holds for any $x \in \mathbb{R}^n$.

Remark

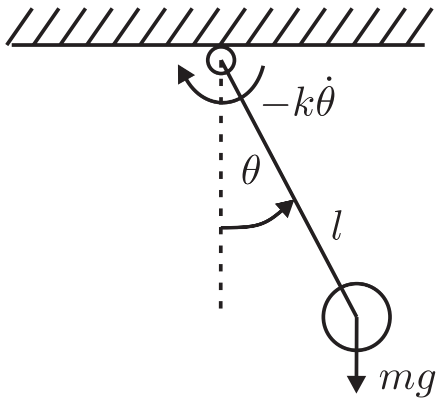
- Strongest stability condition in this lecture
- Implies asymptotic stability

Gap 4

Stability: Example

Pendulum

- Mass m
- Friction torque: $T_\mathrm{f} = -k\dot{\theta}$
- Torque due to gravity: $T_\mathrm{g} = -mgl \sin\theta$
- Acceleration: $M_\mathrm{a} = -ml^2\ddot{\theta}$
- Torque balance: $ml^2\ddot{\theta} = -mgl \sin\theta - k\dot{\theta}$



State Space Model: $x_1 = \theta$ and $x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m\, l^2} x_2$$

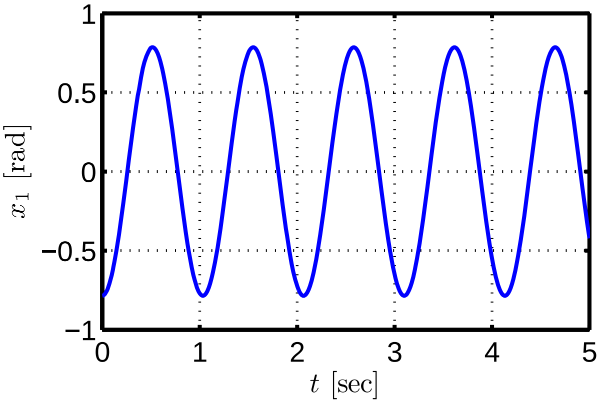
$$\Rightarrow \text{ equilibrium point: } x_1 = x_2 = 0$$

Stability: Example

Simulation:

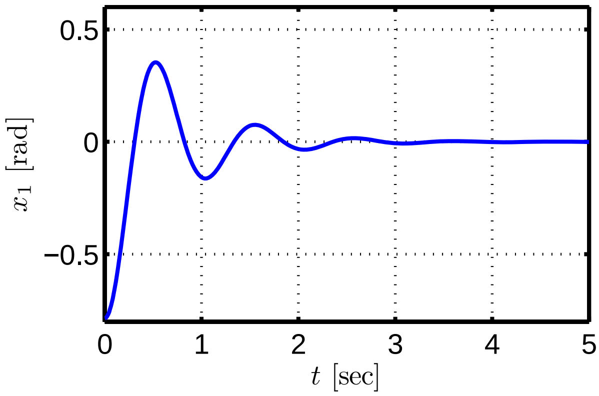
- Initial condition: $x_0 = \begin{bmatrix} -\pi/4 \\ 0 \end{bmatrix}$

No Friction ($k = 0$)



$\Rightarrow$ Stable (not asymptotically)

Friction ($k \neq 0$)



$\Rightarrow$ Asymptotically stable

Stability: Standardization

General Nonlinear Systems

- Multiple equilibrium points
- Each equilibrium point should be analyzed
- We perform stability analysis for standardized equilibrium point
  $\Rightarrow$ Transfer each equilibrium point to the origin $x = 0$

Standardization of Stability Analysis

- Change of variables to move each x_{e} under consideration to the origin
- Consider $\dot{x} = f(x)$ and $f(x_{\mathrm{e}}) = 0$
- Choose new variable $y = x - x_{\mathrm{e}}$
  $\Rightarrow \dot{y} = \dot{x} = f(x) = f(y + x_{\mathrm{e}}) =: g(y)$
  $\Rightarrow g(0) = f(0 + x_{\mathrm{e}}) = 0$

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Stability: Standardization

Standardization of Stability Analysis

- New system equations: $\dot{y} = g(y)$
  $\Rightarrow$ Without loss of generality, we study stability of $\dot{y} = g(y)$ with $y_{\mathrm{e}} = 0$ from now on

Example

Gap 5