Nonlinear Systems and Control

Lecture 4

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Master Course in Electronic and Communication Engineering
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Describing Function Concept: Basic Idea

Goal

- Predict the existence of periodic solutions $y(t) = y(t + 2\pi/\omega)$, $t \in \mathbb{R}$, and their oscillation frequency ω for nonlinear systems

System Model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y) \end{aligned} \qquad (1)$$

- (A, B) is controllable
- (A, C) is observable
- $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is time-invariant nonlinearity

Gap 1

Describing Function Concept: Harmonic Balance

Periodicity Assumption

- Assume periodic solution y represented as Fourier series

$$y(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} a_k \sin k\omega t + b_k \cos k\omega t \quad (a_k, b_k \in \mathbb{R})$$

- Conclude periodicy of $\psi(y(t))$ with same frequency

$$\psi(y(t)) = \frac{d_0}{2} + \sum_{k=1}^{\infty} c_k \sin k\omega t + d_k \cos k\omega t \quad (c_k, d_k \in \mathbb{R})$$

- Generally, c_k, d_k depend on all coefficients a_k, b_k

Gap 2

Describing Function Concept: Output of Nonlinearity

Computation of the First Harmonic

$$\begin{aligned}\psi(a \sin \omega t) &\approx \frac{d_0(a)}{2} + c_1(a) \sin \omega t + d_1(a) \cos \omega t \\ &= \frac{d_0(a)}{2} + \sqrt{c_1(a)^2 + d_1(a)^2} \sin(\omega t + \arctan \frac{d_1(a)}{c_1(a)})\end{aligned}$$

Fourier Coefficients

$$\begin{aligned}d_0(a) &= \frac{1}{2\pi} \int_0^{2\pi} \psi(a \sin \omega t) dt \\ c_1(a) &= \frac{1}{\pi} \int_0^{2\pi} \psi(a \sin \omega t) \sin \omega t d(\omega t) \\ d_1(a) &= \frac{1}{\pi} \int_0^{2\pi} \psi(a \sin \omega t) \cos \omega t d(\omega t)\end{aligned}$$

Describing Function Concept: Definition

Definition (Describing Function)

Consider the system in Equation (1), and assume that the nonlinearity $\psi : \mathbb{R} \to \mathbb{R}$ is skew-symmetric (that is, $d_0(a) = 0$ in the above computation). The describing function $N(a) : \mathbb{R} \to \mathbb{C}$ for ψ is defined as

$$N(a) = \frac{\sqrt{c_1(a)^2 + d_1(a)^2}}{a} e^{j \arctan \frac{d_1(a)}{c_1(a)}} = \frac{c_1(a)}{a} + j \frac{d_1(a)}{a}$$

Discussion of Approximation

- Neglecting all terms in the Fourier series with $k > 1$
- Underlying assumption: $G(s)$ is low-pass filter that attenuates high-frequencies

Gap 3

Describing Function Concept: Relay Example

Gap 4

Describing Function Concept: Relay Example

Gap 5

Describing Function Concept: Hysteresis Example

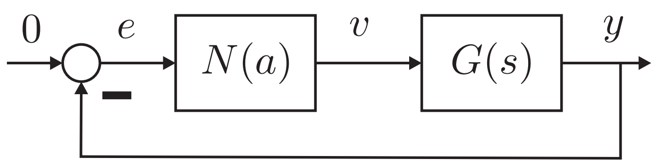
Gap 6

Describing Function Concept: Hysteresis Example

Gap 7

Describing Function Analysis: System Analysis

Harmonic Balance Equation



- $e(t) = a \sin \omega t$
- $v(t) = a|N(a)| \sin(\omega t + \angle N(a))$
- $y(t) = |G(j\omega)||N(a)| a \sin(\omega t + \angle N(a) + \angle G(j\omega))$

Gap 8

$\Rightarrow$ **Condition:** $1 = -G(j\omega)N(a)$ **or** $G(j\omega) = -\frac{1}{N(a)}$

Describing Function Analysis: Analysis Example

Gap 9

Describing Function Analysis: Stability

Possible Properties of Predicted Oscillations

- Decay
- Explode
- Sustained

$\Rightarrow$ Generalized Nyquist plot analysis

Theorem (Nyquist Criterion)

Assume a classical feedback loop with a constant gain K and a transfer function G with no poles on the $j\omega$-axis. Let p be the number of poles of G in the open right half plane and let n be the number of times the $G(j\omega)$-locus encircles the point $-1/K + j0$ in the complex plane (clockwise). Then, all closed-loop poles are in the open left half plane if and only if $p + n \leq 0$.

Describing Function Analysis: Stability

Illustration

Gap 10

Generalized Nyquist Criterion

- Assume that $K \in \mathbb{C}$ (instead of $K \in \mathbb{R}$)
  $\Rightarrow$ Nyquist criterion now applies for encirclement of the complex point $-1/K$

Gap 11

Describing Function Analysis: Stability

Application of Extended Nyquist Criterion to Nonlinear System

Gap 12

Prediction

Gap 13

Describing Function Analysis: Application

Example

Gap 14

Describing Function Analysis: Summary and Remarks

Summary

- Describing function $N(a)$: transfer function between the first harmonic of the input and the output of the nonlinearity
- Analysis: predict periodic solutions of nonlinear systems based on based harmonic balance equation: $1 = -G(j\omega)N(a)$
- Approximation: focus on first harmonic of periodic solutions
- Stability: Prediction based on extended Nyquist analysis

Assumptions

- Skew-symmetric time-invariant nonlinearity
- Focus on first harmonic of periodic solutions (G is low-pass filter)

Remarks

- Predictions can be wrong due to approximation