Nonlinear Systems and Control

Lecture 3

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Master Course in Electronic and Communication Engineering
Credits (2/2/3)

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Second-order Systems: Phase Plane Analysis

State Equations

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = f(x), \quad x(0) = x_0,\ x \in \mathbb{R}^2 \qquad (1)$$

Phase Plane Analysis

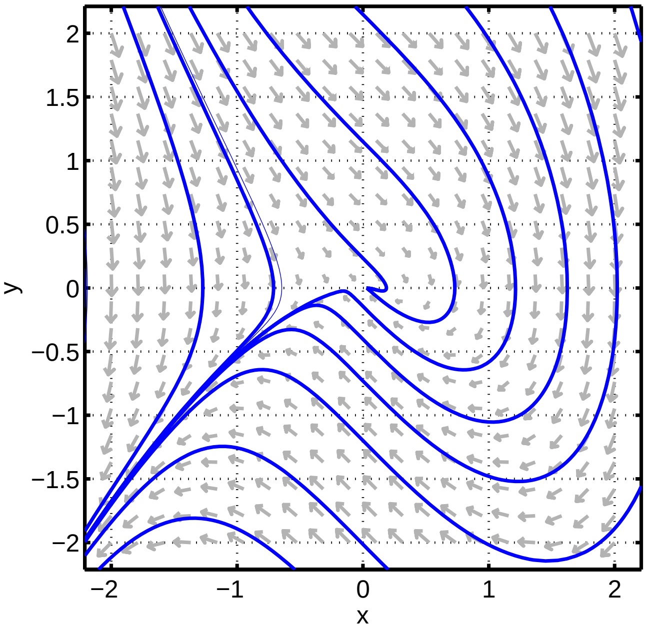
- Visualization of trajectories for various initial conditions in the x_1-x_2 plane
- Assumption: IVP with x_0 has unique solution $x(t)$
- Assign vector with amplitude and direction of $f(x')$ to each point x'
  $\Rightarrow$ Vector $x' + f(x')$ at point x'
  $\Rightarrow$ Vector field diagram for whole x_1-x_2 plane
- Vector field diagram indicates shape of trajectories that pass each point $x' \in \mathbb{R}^2$

Second-order Systems: Phase Plane Analysis

Example

Gap 1

Phase Plane Plot



Second-order Systems: Linear Case

State Equations

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2$$

$\Rightarrow$ Analyze different cases

Different Real Eigenvalues $\lambda_1 \neq \lambda_2 \neq 0$

Gap 2

Second-order Systems: Linear Case

Real Eigenvalues λ_1, λ_2

Gap 3

Complex Eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$

Gap 4

Second-order Systems: Hartman – Grobman Theorem

Linearization

- Consider linearization of $\dot{x} = f(x)$ around x_0

$$A(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}|_{x_0} & \frac{\partial f_1}{\partial x_2}|_{x_0} \\ \frac{\partial f_2}{\partial x_1}|_{x_0} & \frac{\partial f_2}{\partial x_2}|_{x_0} \end{bmatrix}$$

Theorem (Hartman – Grobman Theorem)

Assume that the eigenvalues of $A(x_0)$ are not on the $j\omega$-axis. Then, for a neighborhood $\mathcal{U} \in \mathbb{R}^2$ with $x_0 \in \mathcal{U}$, there exists a continuous map $h : \mathcal{U} \to \mathbb{R}^2$ with a continuous inverse h^{-1} that takes trajectories of the nonlinear system $\dot{x} = f(x)$ onto trajectories of the linear system $\dot{x} = A(x_0)x$.

$\Rightarrow$ If the eigenvalues of $A(x_0)$ do not lie on the imaginary axis, then the type of the equilibrium point x_0 can be deduced from $A(x_0)$

Second-order Systems: Linearization

Example

Gap 5

Second-order Systems: Linearization

Example

Gap 6

Oscillations: Definition

Definition

A nonlinear system is said to oscillate if it has a non-trivial periodic solution, that is, a trajectory $x(t)$ such for some $T > 0$

$$x(t+T) = x(t), \quad \forall t \geq 0$$

Such trajectory $x(t)$ is called a closed orbit and T is called the period.

Special Case: Linear System

$$\dot{x} = Ax$$

- Oscillations if and only if the characteristic polynomial $\det(sI - A)$ has purely imaginary eigenvalues
  $\Rightarrow$ Amplitude depends on initial condition
  $\Rightarrow$ Period depends on the eigenvalue location

Oscillations: Linear System

Example

Gap 7

Practical Limitations

- Purely imaginary eigenvalues are difficult to achieve
- Either eigenvalues with negative real part (damped) or positive real part (instable)
  $\Rightarrow$ Oscillations are not structurally stable

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Oscillations: Limit Cylces for Nonlinear Systems

Example: Van der Pol Oscillator

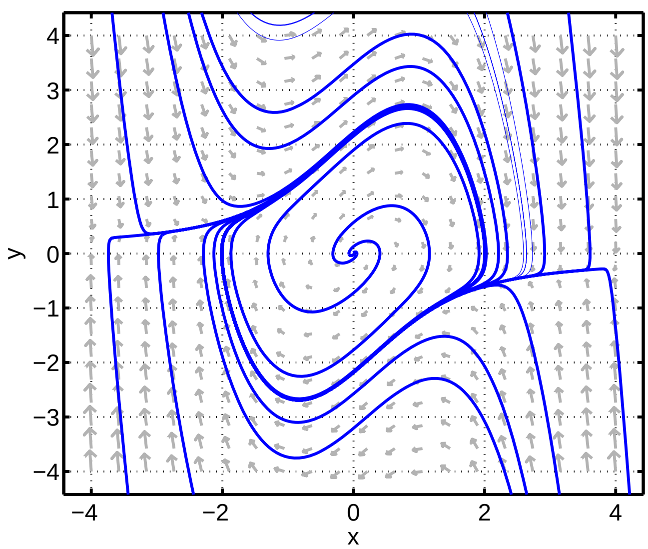
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \mu(1 - x_1^2)x_2, \quad \mu > 0$$

- Appears for example in electrical circuits with vacuum tubes
- Additional dynamics $\mu(1 - x_1^2)x_2$ compared to linear oscillator
  $\Rightarrow$ For $\mu = 0$, the oscillatory linear system is recovered
- Phase plane analysis shows that all system trajectories converge to *limit cycle*
  $\Rightarrow$ Structural stability: convergence to periodic solution in case of deviations from periodic solution

Oscillations: Limit Cylces for Nonlinear Systems

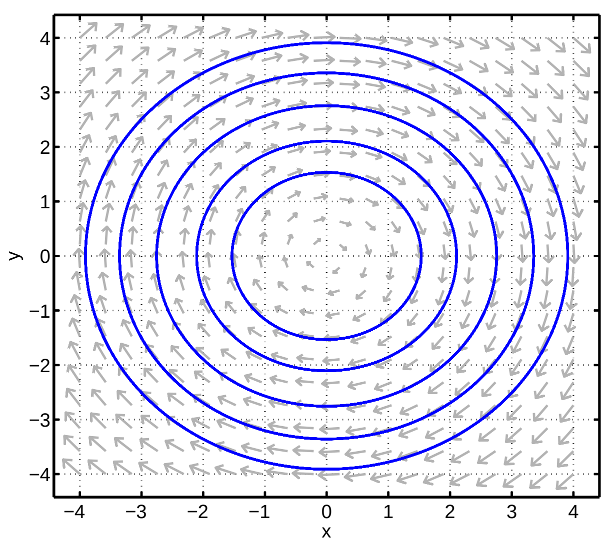
Phase Plane Plot

Van-der Pol Oscillator



$\Rightarrow$ One isolated orbit

Linear System



$\Rightarrow$ Continuum of closed orbits

Oscillations: Properties of Limit Cycles

Stable Limit Cycle

- All trajectories converge toward the limit cycle
  $\Rightarrow$ Van der Pol oscillator

Instable Limit Cycle

- All trajectories diverge from the limit cycle
  $\Rightarrow$ Van der Pol oscillator equation for $\mu < 0$

Semi-stable Limit Cycle

- All trajectories on one side of the limit cycle (inside or outside) converge to the limit cycle while all trajectories on the other side of the limit cycle diverge from the limit cycle

Oscillations: Poincaré–Bendixson Theorem

Theorem

Let $\dot{x} = f(x)$ be a <u>two-dimensional</u> autonomous system with a continuously differentiable f in the domain $\mathcal{D} \subseteq \mathbb{R}^2$ and assume that

1. *$\mathcal{R} \subseteq \mathcal{D}$ is a closed and bounded set without equilibrium points of $\dot{x} = f(x)$*
2. *There is a trajectory $x(t)$ that is confined to $\mathcal{R}$*

Then, either $\mathcal{R}$ is a closed orbit or $x(t)$ converges to a closed orbit

$\Rightarrow$ For two-dimensional systems, each trajectory in a bounded region without equilibrium points converges to a limit cycle

Remarks

- There is no such statement for systems with dimension > 2
- Higher-dimensional systems show new phenomena (see Marquez)