Nonlinear Systems and Control

Lecture 2

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Master Course in Electronic and Communication Engineering
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Solution of Differential Equations: Nonlinear Systems

General Model

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u)$$

- State vector: $x \in \mathbb{R}^n$; input: $u \in \mathbb{R}^m$; output: $y \in \mathbb{R}^p$
- Right-hand side (rhs): $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$
- Output function: $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$
- n: state space order; m: number of inputs; p number of outputs

Convention

- Autonomous system: $\dot{x} = f(x)$
- Non-autonomous system: $\dot{x} = f(t, x)$
- Autonomous system with inputs: $\dot{x} = f(x, u)$

Solution of Differential Equations: Definition

Definition

Consider a non-autonomous system

$$\dot{x} = f(t, x). \quad (1)$$

For an interval $\mathcal{I} \subseteq \mathbb{R}$, the time function $x : t \mapsto x(t) : \mathcal{I} \to \mathbb{R}^n$ is a *solution* of (1) if x is differentiable on $\mathcal{I}$ and fulfills (1) (that is, $\dot{x}(t) = f(t, x(t))$ for all $t \in \mathcal{I}$).

If (1) has an initial condition $x(t_0) = x_0$, then the combination of (1) and $x(t_0) = x_0$ is denoted as an *initial value problem* (IVP).

Remark: Direct solution for linear time-invariant systems

$$\dot{x} = f(t, x) = A\,x, \quad x(0) = x_0 \Rightarrow x(t) = e^{At}\,x_0$$

Solution of Differential Equations: Illustration

Example

$$\dot{x} = \frac{1}{x}, \quad x(0) = 1$$

Gap 1

Solution of Differential Equations: Existence

Theorem (Peano)

Consider the non-autonomous system in (1). If f is continuous in a neighborhood $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}^n$ of (t_0, x_0), then there is at least one solution of the IVP.

Remark

- Neighborhood can be very small

Example

Gap 2

Solution of Differential Equations: Existence

Example

Gap 3

Solution of Differential Equations: Lipschitz-continuity

Definition (Lipschitz-continuity)

A function $f : (t, x) \mapsto f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be *locally Lipschitz-continuous* with respect to x in the region $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}^n$ if there is a constant $L > 0$ (Lipschitz-constant) such that

$$||f(t, \hat{x}) - f(t, \tilde{x})|| \leq L||\hat{x} - \tilde{x}||, \forall (t, \hat{x}), (t, \tilde{x}) \in \mathcal{R}$$

If $\mathcal{R} = \mathbb{R} \times \mathbb{R}^n$, then f is *globally Lipschitz-continuous*.

Example

Gap 4

Solution of Differential Equations: Lipschitz-continuity

Example

Gap 5

Remark: Sufficient conditions for Lipschitz-continuity

- If $\frac{\partial f}{\partial x}$ is continuous in x, then f is locally Lipschitz-continuous
- If $\frac{\partial f}{\partial x}$ is continuous and bounded on $\mathbb{R} \times \mathbb{R}^n$, then f is globally Lipschitz-continuous

Solution of Differential Equations: Uniqueness

Theorem (Picard-Lindelöf)

Consider a non-autonomous system with initial condition $x(t_0) = x_0$ and let $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}^n$ with $(t_0, x_0) \in \mathcal{R}$. If f is continuous in t on $\mathcal{R}$ and f is Lipschitz-continuous in x on $\mathcal{R}$, then there is an interval $\mathcal{I} \subseteq \mathbb{R}$ with $t_0 \in \mathcal{I}$ such that the IVP has a unique solution on $\mathcal{I}$.

Remark

- If f is globally Lipschitz-continuous
  $\Rightarrow$ There is a unique solution of the IVP for all times
- If f is locally Lipschitz-continuous everywhere in $\mathbb{R} \times \mathbb{R}^n$
  $\Rightarrow$ Either there is a unique solution of the IVP for all times or IVP has finite escape time (see example)

Solution of Differential Equations: Uniqueness

Example

Gap 6

Equilibrium Points: Definition

Definition

Let $\dot{x} = f(t, x)$ be a non-autonomous system that is defined over a region $\mathcal{R} \in \mathbb{R} \times \mathbb{R}^n$. A point $x = x_{\mathrm{e}} \in \mathcal{R}$ is called an *equilibrium point* of the system if $f(x_{\mathrm{e}}) = 0$.

⇒ If the system state is x_{e}, then it remains at x_{e} for all future times

⇒ Practical interpretation: an equilibrium point could for example be a set-point for set-point control

Example

Gap 7

Equilibrium Points: Comments

Example

Gap 8

Nonlinear Systems expected in this Lecture

- Unique solution that exists for all times
- Potentially multiple equilibrium points

First-order System Analysis: Properties

State Equation

$$\dot{x} = f(x), \quad x(0) = x_0, \; x \in \mathbb{R} \tag{2}$$

- Equilibrium points: $f(x_e) = 0$
- Solution $x(t)$ of (2) is denoted as *trajectory*

Behavior Around Equilibrium Points

- Graphical analysis: plot $\dot{x}$ over x
  $\Rightarrow$ Study value of derivative $\dot{x}$ around each equilibrium point x_e

Gap 9

First-order Systems: Observations

Gap 10

General Observation

Dynamics of first-order system is dominated by behavior around equilibrium points

- Either trajectories approach equilibrium point
- Or trajectories diverge from equilibrium point
  $\Rightarrow$ No oscillations for first-order systems
- Generalization of result that is known for linear systems