Nonlinear Systems and Control

Lecture 12

Associate Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineering – Çankaya University

Master Course in Electronic and Communication Engineering
Credits (3/0/3)

Webpage: http://ECE564.cankaya.edu.tr

Input/Output Linearization: State Transformation

Theorem (State Transformation)

Assume that the nonlinear system has a relative degree of $r \leq n$ around the point $\overline{x} \in \mathcal{D}$. Then, there exist functions $t_{r+1}(x), \ldots, t_n(x)$ such that

$$z = t(x) := \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ t_{r+1}(x) \\ \vdots \\ t_n(x) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix}$$

is a local diffeomorphism around $\overline{x}$, that is, $t(x)$ is continuously differentiable and its inverse t^{-1} uniquely exists on $\mathcal{D}$ and is differentiable. $t_{r+1}(x), \ldots, t_n(x)$ can be chosen such that $L_g t_{r+1} = \cdots = L_g t_n = 0$.

Input/Output Linearization: Helicopter Example

Example

Gap 1

Flatness Set-point Control Feedforward Control Input/Output Linearization

Input/Output Linearization: Helicopter Example

Computation

Gap 2

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Input/Output Linearization: Helicopter Example

Computation

Gap 3

Flatness: Recall

Byrnes-Isidori Canonical Form (BIC)

$$\left.\begin{aligned}\dot{z}_1 &= z_2\\ &\vdots\\ \dot{z}_r &= b(z) + a(z)u\end{aligned}\right\}\text{ Chain of integrators}$$

$$\left.\begin{aligned}\dot{\eta}_1 &= q_1(z)\\ &\vdots\\ \dot{\eta}_{n-r} &= q_{n-r}(z)\end{aligned}\right\}\text{ internal dynamics}$$

$$y = z_1$$

⇒ Linearizing state feedback:

$$u = \frac{1}{a(z)}(w - b(z))$$

⇒ No direct input for internal dynamics

Block Diagram

Gap 4

Flatness: Definition

Special Case of the BIC: $n = r$

$$\dot{z}_1 = z_2$$
$$\vdots$$
$$\dot{z}_n = b(z) + a(z)u$$
$$y = z_1$$

Block Diagram

Gap 5

Remarks

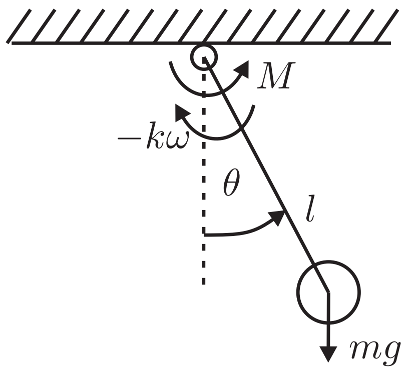
- No internal dynamics
- The system is called flat and $y_f = z_1$ is the *flat output* of the system
- The system can be converted in a pure chain of integrators by the state feedback $u = \frac{1}{a(z)}(v - b(z))$

Flatness: Pendulum Example

State Equations

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{l^2}\sin(\theta) - k\omega \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \end{bmatrix} M$$
$$y = \theta - \frac{\pi}{2}$$

- Angle θ, length l, input torque M



Gap 6

Flatness: Set-point Control

Stabilizing State Feedback

- Linearizing state feedback: $u = \dfrac{v - b(z)}{a(z)}$

  $\Rightarrow y^{(n)} = b(z) + a(z)\dfrac{1}{a(z)}(v - b(z)) = v$
- Linear state feedback: $v = -k_1 z_1 - k_2 z_2 - \cdots - k_n z_n$ for $k_i \in \mathbb{R}$

  $\Rightarrow \dot{z}_n = y^{(n)} = v = -k_0 z_1 - k_1 z_2 - \cdots k_{n-1} z_n$
- Linear differential equation in y

  $\Rightarrow y^{(n)} + k_{n-1} y^{(n-1)} + \cdots + k_1 \dot{y} + k_0 y = 0$

$\Rightarrow$ Asymptotically stable equilibrium point at $z = 0$ if the polynomial $s^n + k_{n-1} s^{n-1} + \cdots + k_1 s + k_0$ is Hurwitz

Flatness: Set-point Control

Block Diagram

Gap 7

Flatness: Set-point Control

Stabilizing Feedback in z-Coordinates

$$u = \frac{1}{a(z)}(v - b(z)) = \frac{1}{a(z)}(-k_{n-1}z_n - \cdots - k_1 z_2 - k_0 z_1 - b(z))$$

Stabilizing Feedback in Original Coordinates

- Recall $x = t^{-1}(z)$ and $z_i = L_f^{i-1}h(x)$

$$u = \frac{1}{a(t^{-1}(z))}(-k_{n-1}L_f^{n-1}h(x) - \cdots - k_1 L_f h(x) - k_0 h(x) - b(t^{-1}(z)))$$

$\Rightarrow$ **Asymptotically stable equilibrium point at** $x = t^{-1}(0)$

Remarks

- Linear system behavior in z-coordinates
- Nonlinear system behavior in x-coordinates!

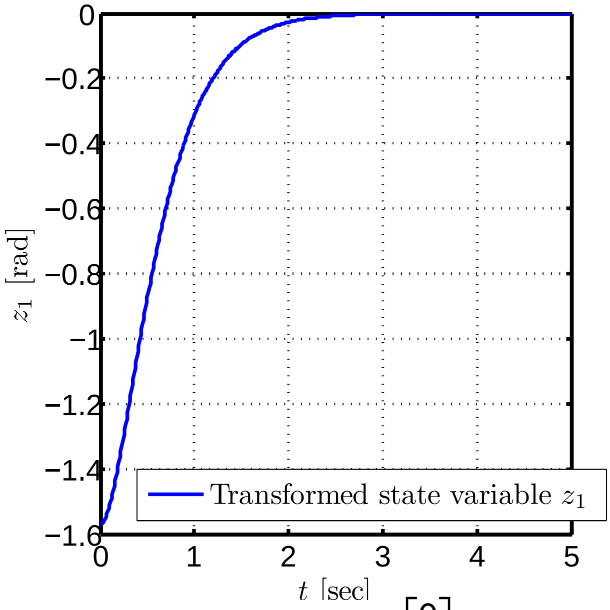
Flatness: Set-point Control Example

Pendulum

Gap 8

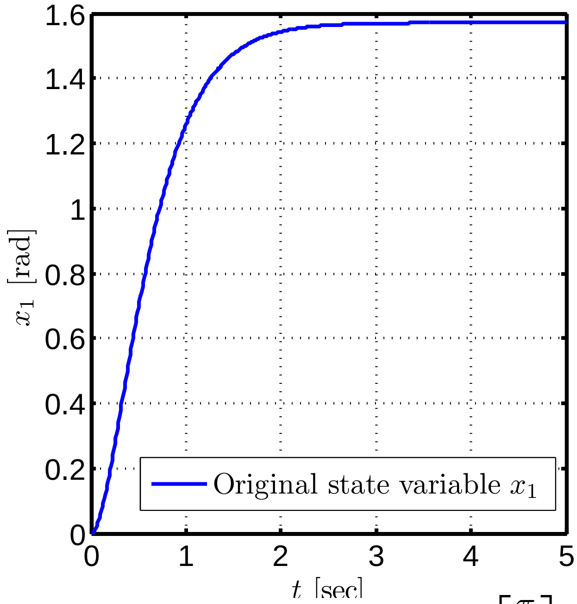
Flatness: Set-point Control Example

Simulation: z-Coordinates



$\Rightarrow$ Stabilization of $z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Simulation: Original Coordinates



$\Rightarrow$ Stabilization of $t^{-1}(z) = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$

Feedforward Control: Properties

Recall from Input/Output Linearization

- $z = t(x)$, $x = t^{-1}(z)$ with $t(0) = 0$ and also $t^{-1}(0) = 0$
- $y_f = z_1 = h(x)$, $\dot{y}_f = L_f h = z_2$, $\ddot{y}_f = L_f^2 h = z_3$, ...

Properties in Case of Flatness

- $x = t^{-1}(z) = \varphi_x(y_f, \dot{y}_f, \ldots, y_f^{(n-1)})$

  $\Rightarrow$ x can be expressed in terms of $n-1$ flat output derivatives
- $u = \varphi_u(y_f, \dot{y}_f, \ldots, y_f^{(n)})$

  $\Rightarrow$ u can be expressed in terms of n flat output derivatives

Gap 9

Feedforward Control: Trajectory Tracking

Gap 10

Trajectory Tracking

- Assume given desired trajectory $y_d(t)$ for system output
  $\Rightarrow$ Input $u = \varphi_u(y_d, \dot{y}_d, \dots, y_d^{(n)})$ ensures that system tracks $y_d(t)$

Feedforward Architecture

Gap 11

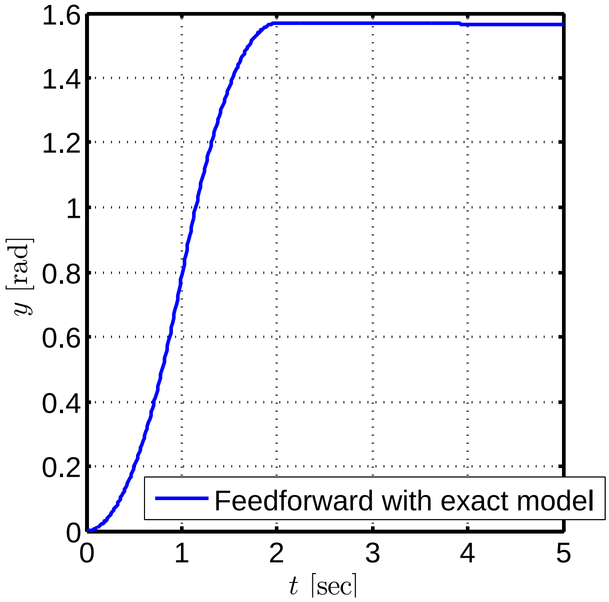
Feedforward Control: Example

Pendulum

Gap 12

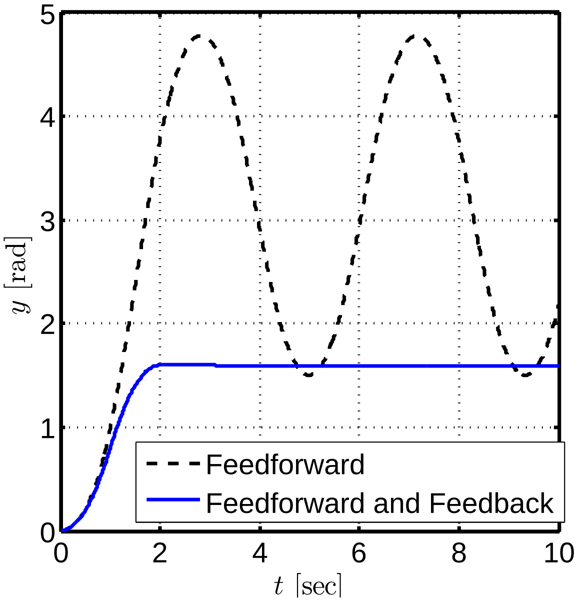
Feedforward Control: Example

Simulation with Exact Model



$\Rightarrow$ Exact tracking as designed

Simulation with Uncertain Model



$\Rightarrow$ Feedback is essential in case of uncertainties/disturbances

Feedforward Control: Combination with Feedback

Feedforward Architecture with Feedback

Gap 13

Stabilization of Output Error

$$\underbrace{y^{(n)} - y_d^{(n)}}_{e^{(n)}} = b(z) - a(z)u - y_d^{(n)} = b(z) - a(z)\left(\frac{y_d^{(n)} - b(z) + v}{a(z)}\right) - y_d^{(n)} = v$$

Choose $v = -k_{n-1}\overbrace{(y^{(n-1)} - y_d^{(n-1)})}^{e^{(n-1)}} - \cdots - k_1\overbrace{(\dot{y} - \dot{y}_d)}^{\dot{e}} - k_0\overbrace{(y - y_d)}^{e}$

$\Rightarrow$ Stable error dynamics: $e^{(n)} - k_{n-1}e^{(n-1)} - \cdots - k_1\dot{e} - k_0 e = 0$ if $s^n + k_{n-1}s^{n-1} + \cdots + k_1\dot{s} + k_0$ is Hurwitz

Input/Output Linearization: General Setup

Relative Degree $r < n$

$$\dot{z}_1 = z_2$$
$$\vdots$$
$$\dot{z}_r = b(z) + a(z)u$$
$$\dot{\eta} = q(z)$$
$$y = z_1$$

Helicopter Example ($r = 2$)

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -g \tan z_3 - \frac{1}{M \cos z_3} u$$
$$\dot{z}_3 = M \cos z_3 z_4 - L z_2$$
$$\dot{z}_4 = -lg \tan z_3 - M \cos z_3 (z_4 - L z_2)^2 \tan z_3$$
$$y = z_1$$

Input/Output Relation

- Input/output behavior can be written as $y^{(r)} = b(z) + a(z)u$
  $\Rightarrow$ Analogous to case of flatness
- Internal dynamics $\dot{\eta} = q(z)$ cannot be controlled by input u
  $\Rightarrow$ Stability analysis of internal dynamics is required

Flatness Set-point Control Feedforward Control Input/Output Linearization

Input/Output Linearization: Set-point Control

Input/Output Behavior

- Asymptotic stabilization of $z_1 = \cdots = z_r = 0$ analogous to flatness

$$\Rightarrow u = \frac{1}{a(z)}(-k_0 z_1 - k_1 z_2 - \cdots - k_{r-1} z_r - b(z))$$

Analysis of Internal Dynamics

- Write internal dynamics as $\dot{\eta} = q(z) = q(z_1, \ldots, z_r, \eta)$
- For asymptotic stabilization of $z_1 = \cdots = z_r = 0$
  $\Rightarrow$ Null-dynamics: $\dot{\eta} = q(0, \ldots, 0, \eta)$

Asymptotically Stable Set-point Control

- Use asymptotically stabilizing feedback law for input/output behavior
- Verify asymptotic stability of null-dynamics
  $\Rightarrow$ Use methods discussed in the lecture (linearization, Lyapunov stability and extensions, passivity)

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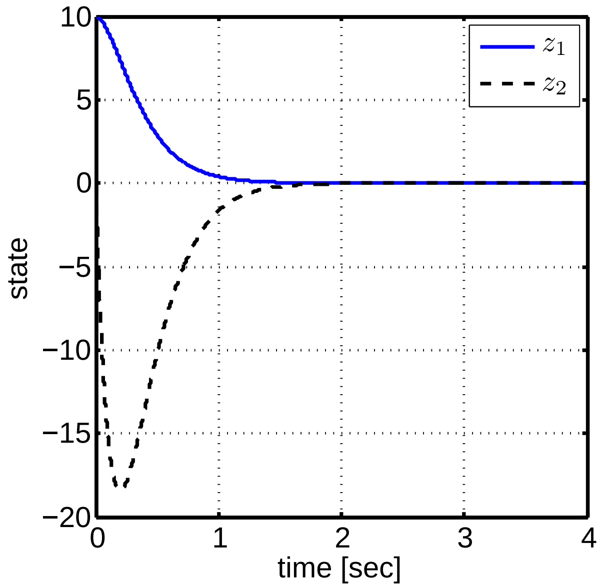
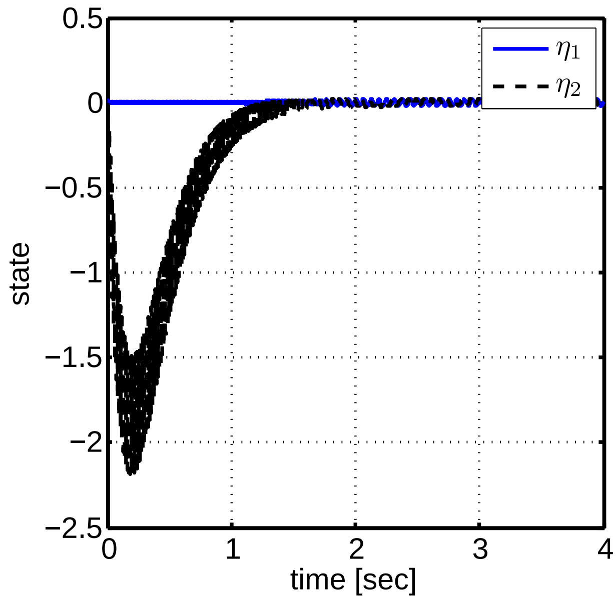
Input/Output Linearization: Helicopter Example

Input/output Behavior



⇒ Asymptotic stability of $z_1 = z_2 = 0$

Internal Dynamics



⇒ Stable internal dynamics (but not asymptotically stable)

Input/Output Linearization: Feedforward

Input/output Behavior

- Feedforward computation analogous to flatness
  $$\Rightarrow u = \varphi(y_d, \dot{y}_d, \ldots, y_d^{(r)})$$

Analysis of Internal Dynamics

- Consider y_d and its derivatives as input to the internal dynamics
- Write internal dynamics as $\dot{\eta} = q(y_d, \dot{y}_d, \ldots, y_d^{(r-1)}, \eta)$
  ⇒ Stability if internal dynamics are input-to-state stable (see Khalil's book for exact definition)
- Verification of input-to-state stability usually difficult
  ⇒ Perform simulation of internal dynamics to verify system stability