Exercise Sheet 6: Lyapunov Stability

Problem 10:

The following Lyapunov functions $V(x)$ and their time derivatives $\dot{V}(x)$ are given for some nonlinear system that has an equilibrium point at the origin. Classify the properties of the equilibrium point as (i) stable, (ii) asymptotically stable, (iii) inconclusive information.

a. $V(x) = x_1^2 + x_2^2$ and $\dot{V}(x) = -x_1^2$

b. $V(x) = (x_1^2 + x_2^2 - 1)$ and $\dot{V}(x) = -(x_1^2 + x_2^2)$

c. $V(x) = 7\,x_1^2 + 22\,x_2^2$ and $\dot{V}(x) = -(x_1^2 + x_2^2)$

d. $V(x) = (x_1^2 + 2x_1x_2 + x_2^2)$ and $\dot{V}(x) = -(x_1 + x_2)^2$

Problem 11:

We are given the following nonlinear system

$$\dot{x}_1 = x_2 + \beta(x_1^3/3 - x_1)$$
$$\dot{x}_2 = -x_1$$

Investigate the stability of the equilibrium point $x = 0$ for the following cases

a. $\beta > 0$

b. $\beta = 0$

Hint: Use the Lyapunov canditate $V(x) = x_1^2 + x_2^2$ whenever appropriate.

Problem 12:

We are given the following nonlinear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - \mathrm{sat}(2x_1 + x_2),$$

where $\mathrm{sat}(\cdot)$ denotes the saturation function

$$\mathrm{sat}(v) = \begin{cases} 1 & \text{if } v \geq 1 \\ -1 & \text{if } v \leq -1 \\ v & \text{otherwise} \end{cases}$$

a. Show that the origin is stable

b. Show that all trajectories starting in the first quadrant to the right of the curve $x_1x_2 = c$ (with sufficiently large $c > 0$) cannot reach the origin

Hint: Consider $V(x) = x_1x_2$, calculate $\dot{V}(x)$ and show that on the curve $x_1x_2 = c$, the derivative $\dot{V}(x) > 0$ when c is large enough

c. An equilibrium point is called "globally asymptotically stable" if it is stable and all trajectories converge to the equilibrium point. Show that the origin is not globally asymptotically stable

Problem 13:

Consider the following nonlinear system:

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2 - x_2\,\mathrm{sat}(x_2^2 - x_3^2)$$
$$\dot{x}_3 = x_3\,\mathrm{sat}(x_2^2 - x_3^2)$$

a. Show that the origin is the unique equilibrium point of the nonlinear system

b. Use the candidate function $V(x) = x_1^2 + x_2^2 + x_3^2$ to prove global asymptotic stability of the nonlinear system